2 Frobenius Series Solution of Ordinary Differential Equations

At the start of the differential equation section of the 1B21 course last year, you met the linear first-order separable equation

$$\left(\frac{dy}{dx}\right) = \alpha\, y\,, \tag{2.1}$$

where α is a constant. You were also shown how to integrate the equation to get the solution

$$y = A\, e^{\alpha\, x}\,, \tag{2.2}$$

where A is an arbitrary integration constant. The solution can be expanded in a power series in x and I want to show explicitly that this power series does indeed satisfy Eq. (2.1):

$$\begin{aligned} y &= A\left[1+\alpha x+\frac{1}{2}\alpha^2x^2+\frac{1}{6}\alpha^3x^3+\cdots+\frac{1}{(n-1)!}\alpha^{n-1}x^{n-1}+\frac{1}{n!}\alpha^n x^n+\cdots\right] \\ \frac{dy}{dx} &= A\left[0+\alpha \;+\; \alpha^2 x \;+\frac{1}{2}\alpha^3x^2+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{1}{(n-1)!}\alpha^n x^{n-1}+\cdots\right]. \end{aligned}$$

Notice that, although similar terms are pushed one to the right, y' is clearly equal to $\alpha\, y$.

Let us now see how this is handled using the summation notation. Here

$$y = A\sum_{n=0}^{\infty}\frac{1}{n!}\,\alpha^n\, x^n\,, \tag{2.3}$$

$$\frac{dy}{dx} = A\sum_{n=0}^{\infty}\frac{1}{(n-1)!}\,\alpha^n\, x^{n-1} = A\,\alpha\sum_{n=0}^{\infty}\frac{1}{(n-1)!}\,\alpha^{n-1}\, x^{n-1}\,. \tag{2.4}$$

Eq. (2.4) does not yet look quite like α times Eq. (2.3) because one sees x^{n-1} rather than x^n. This is a reflection of all the terms being pushed one over, as noted in the long-hand representation above. To show the equivalence, first note that the $n = 0$ term is actually absent from the series for y' and so the series effectively starts at $n = 1$. Secondly, the n in the series is a dummy variable, like an integration variable, which does not occur in the final answer. We can therefore make the substitution $m = n - 1$ in Eq. (2.4) to find

$$\frac{dy}{dx} = A\,\alpha\sum_{m=0}^{\infty}\frac{1}{m!}\,\alpha^m\, x^m\,, \tag{2.5}$$

which is exactly $\alpha\, y$.

Generally the boot is on the other foot. We often end up with a differential equation (*e.g.* the Legendre equation) which we cannot solve by inspection as we have done here. We want then to develop techniques for finding directly series solutions for differential equations.

Let us try for a solution of Eq. (2.1) in the form

$$y = x^k\sum_{n=0}^{\infty}a_n\, x^n = \sum_{n=0}^{\infty}a_n\, x^{n+k}\,, \tag{2.6}$$

where $a_0 \neq 0$. Thus the value of the index k is defined by the condition that the first term a_0 doesn't vanish. Differentiating term-by-term leads to

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+k)\, a_n\, x^{n+k-1}\,. \tag{2.7}$$

Inserting Eqs. (2.6) and (2.7) into the differential equation, we find

$$\sum_{m=0}^{\infty}(m+k)\, a_m\, x^{m+k-1} = \alpha\sum_{n=0}^{\infty}a_n\, x^{n+k} \tag{2.8}$$

where, to avoid confusion, I have used a different dummy variable m on the left from the n on the right. Since this equation has to be true for a range of values of x around the origin, it must be true separately for each power of x. We must therefore compare the coefficients of different powers of x on the two sides of Eq. (2.8). To facilitate the comparison, let $m = n+1$ on the left hand side.

$$\sum_{n=-1}^{\infty} (n+k+1)\, a_{n+1}\, x^{n+k} = \alpha \sum_{n=0}^{\infty} a_n\, x^{n+k} \,. \tag{2.9}$$

Since we now have powers x^{n+k} explicitly on both sides of the equation, it is simple to deduce that $(n+k+1)\, a_{n+1} = \alpha\, a_n$, *i.e.*

$$a_{n+1} = \frac{\alpha}{n+k+1}\, a_n \,. \tag{2.10}$$

This is a very simple example of a recurrence relation, which allows us to evaluate all the higher coefficients from the first one. It does however contain the unknown index k. How is its value fixed?

The lowest power of x on the left hand side of Eq. (2.9) is $a_0\, k\, x^{k-1}$ (corresponding to $n = -1$), and there is nothing like this on the right hand side because the sum there starts with $n = 0$. The term must therefore be made to cancel

$$a_0\, k = 0 \,. \tag{2.11}$$

Since $a_0 \neq 0$, this can only happen if $k = 0$. Eq. (2.11) is a very simple example of what is called an indicial equation; it fixes the index k.

With $k = 0$, the recurrence relation becomes

$$a_{n+1} = \frac{\alpha}{n+1}\, a_n \,. \tag{2.12}$$

To use the same notation as in the previous solution, let $a_0 = A$. Then

$$\begin{aligned} a_1 &= \alpha A \,, \\ a_2 &= \frac{\alpha}{2}\, a_1 = \frac{\alpha^2}{2}\, A \,, \\ a_3 &= \frac{\alpha^3}{6}\, A \qquad etc. \end{aligned}$$

In general

$$a_{n+1} = \frac{\alpha}{n+1} \times \frac{\alpha}{n} \times \frac{\alpha}{n-1} \times \cdots \times \frac{\alpha}{1} \times A = \frac{\alpha^{n+1}}{(n+1)!}\, A \,. \tag{2.13}$$

2.1 Possible problems

Let me give you a couple of examples to compare.

Example 1

Take first the case of

$$\frac{dy}{dx} = \frac{\alpha\, y}{x} \,.$$

The right hand side blows up at $x = 0$ but not too badly. In the notation that we shall use later, there is a regular singularity at $x = 0$. As before we try for a solution of the form

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k} \,, \quad \text{hence} \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1} \,, \\ \frac{\alpha\, y}{x} &= \sum_{n=0}^{\infty} \alpha\, a_n\, x^{n+k-1} \,. \end{aligned}$$

Looking first at the lowest term, corresponding to $n = 0$, we see that

$$a_0(\alpha - k)\, x^{k-1} = 0\,.$$

But, since $a_0 \neq 0$, the only solution is $k = \alpha$.

Examining now the higher terms, we have to satisfy

$$a_n(\alpha - k - n) = n\, a_n = 0 \quad \text{for} \quad n > 0\,.$$

This is only possible if $a_n = 0$ for $n \geq 1$, so that the total solution is $y = a_0\, x^\alpha$.

Example 2

Contrast this with the case of

$$\frac{dy}{dx} = \frac{\alpha\, y}{x^2}\;,$$

where the right hand side blows up a bit faster at $x = 0$. We call this an irregular singularity. The solution is

$$y = A\, e^{-\alpha/x}\;,$$

which has an essential singularity at $x = 0$ and for which no power series in x is possible in this region. How does this manifest itself in the Frobenius method?

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\;, \quad \text{hence} \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1}\;, \\ \frac{\alpha\, y}{x^2} &= \sum_{n=0}^{\infty} \alpha\, a_n\, x^{n+k-2}\;. \end{aligned}$$

The lowest power of x is x^{k-2} and this is multiplied by $\alpha\, a_0$. Provided that $\alpha \neq 0$, the only solution would require $a_0 = 0$. But the value of k was determined by requiring that $a_0 \neq 0$. These two conditions are in mutual contradiction and so there is no power series solution in x.

Example 3

Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{1-x}\;,$$

which has solution $y = A/(1-x)$. By the series method we would obtain

$$\sum_{n=0}^{\infty} (1-x)\, a_n\, (n+k)\, x^{n+k-1} = \sum_{n=0}^{\infty} a_n\, x^{n+k}\;.$$

Multiplying out by the $-x$ factor and taking it over to the other side,

$$\sum_{n=0}^{\infty} a_n\, (n+k)\, x^{n+k-1} = \sum_{n=0}^{\infty} a_n\, (n+k+1)\, x^{n+k}\;.$$

Now change the dummy index on the left by letting $n \to n+1$, to give

$$\sum_{n=-1}^{\infty} a_{n+1}\, (n+k+1)\, x^{n+k} = \sum_{n=0}^{\infty} a_n\, (n+k+1)\, x^{n+k}\;.$$

The indicial equation is obtained by demanding that the lowest power, corresponding to $n = -1$ on the left hand side, should vanish. This requires $k = 0$. The recurrence relation is then

$$a_{n+1} = a_n\;,$$

which means that all the coefficients are equal to a_0. The full solution is therefore

$$y = a_0 \sum_{n=0}^{\infty} x^n \,.$$

This solution is fine provided that the series converges and you saw in the 1B21 course that this geometric series diverges if $\mid x \mid > 1$. This is not entirely unexpected because the equation has a regular singular point at $x = 1$. In summary:

a: You cannot make any power series expansion about an irregular point, *i.e.* one where y' diverges faster than $1/x$.

b: The power series may not converge if x is too large.

Both these problems are present in second order equations, to which we now turn.

2.2 Second Order Equations

In the 1B21 course you solved the simple harmonic oscillator equation

$$\frac{d^2y}{dx^2} + \omega^2\, y = 0\,. \tag{2.14}$$

The most general solution is

$$y = A\,\cos\omega x + B\,\sin\omega x\,, \tag{2.15}$$

where A and B are arbitrary constants to be fixed by the boundary conditions. A second order linear equation has two arbitrary constants.

Note that the two individual solutions of this equation, *viz* $\cos\omega x$ and $\sin\omega x$, are respectively even and odd functions of the independent variable x. Why is this? First write down Eq. (2.14) for a function $y = f(x)$ and then let $x \to -x$. We then have the two equations

$$\begin{aligned} \frac{d^2 f(x)}{dx^2} + \omega^2\, f(x) &= 0\,, \\ \frac{d^2 f(-x)}{dx^2} + \omega^2\, f(-x) &= 0\,. \end{aligned} \tag{2.16}$$

Thus $f(-x)$ satisfies the same equation as $f(x)$ and this is because all the operators in Eq. (2.14) are even; $\dfrac{d^2}{dx^2}$ doesn't change when $x \to -x$. Any linear combinations of $f(x)$ and $f(-x)$ also satisfy the equations. In particular, the even and odd combinations

$$f_e(x) = \tfrac{1}{2}[f(x) + f(-x)]\,, \tag{2.17}$$
$$f_o(x) = \tfrac{1}{2}[f(x) - f(-x)] \tag{2.18}$$

also satisfy the equation. This is the real reason why $\cos\omega x$ and $\sin\omega x$ are solutions to the oscillator equation. Of course, this argument does not say that the basic solutions have to be either even or odd, but one can always choose them so to be. We will use this argument when we study the Legendre equation in detail.

Try now for a series solution of the oscillator equation;

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1}\,, \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2}\,. \end{aligned} \tag{2.19}$$

Inserting these into Eq. (2.14), we find that

$$\sum_{n=0}^{\infty}(n+k)(n+k-1)\,a_n\,x^{n+k-2}+\omega^2\sum_{n=0}^{\infty}a_n\,x^{n+k}=0\,. \tag{2.20}$$

The indicial equation is obtained by looking at the coefficient of the lowest power here, *viz* x^{k-2}. Since this only occurs in the first sum (for $n=0$), it must vanish:

$$k(k-1)\,a_0\,x^{k-2}=0\,.$$

There are therefore two possible values of the index, $k=0$ or $k=1$, and this is quite typical for a second order equation.

Changing the dummy index in the first sum by $n\to n+2$,

$$\sum_{n=-2}^{\infty}(n+k+2)(n+k+1)\,a_{n+2}\,x^{n+k}+\omega^2\sum_{n=0}^{\infty}a_n\,x^{n+k}=0\,, \tag{2.21}$$

all the powers now look the same and so we can compare coefficients to obtain the recurrence relation

$$(n+k+2)(n+k+1)\,a_{n+2}+\omega^2\,a_n=0\,, \tag{2.22}$$

which means that

$$a_{n+2}=-\frac{\omega^2}{(n+k+2)(n+k+1)}\,a_n\,. \tag{2.23}$$

Given the value of a_0, this allows us to evaluate a_2, and then a_4 *etc.* The odd a_n are completely independent and, as far as getting a solution is concerned, we can put them all to zero. This independence of the odd and even a_n is a consequence of the fact that odd and even solutions of the differential equation are possible. It therefore follows from the fact that the differential operator is even in x, as shown by Eq. (2.16). In order to generate these purely odd/even solutions, it is easiest to put $a_1=0$. If we don't, we do not create extra solutions; we merely mix some of the $k=1$ solution into that with $k=0$.

Solution for $k=0$

The recurrence relation

$$a_{n+2}=-\frac{\omega^2}{(n+2)(n+1)}\,a_n\,. \tag{2.24}$$

has the solution

$$\begin{aligned} a_n &= (-\omega^2)^{n/2}\,a_0/n! && (n\text{ even})\,,\\ &= 0 && (n\text{ odd})\,. \end{aligned} \tag{2.25}$$

The total solution is then

$$y=a_0\sum_{n\ even}(-1)^{n/2}(\omega x)^n\,\frac{1}{n!}=a_0\,\cos\omega x\,. \tag{2.26}$$

Solution for $k=1$

The recurrence relation is

$$a_{n+2}=-\frac{\omega^2}{(n+3)(n+2)}\,a_n\,, \tag{2.27}$$

so that

$$\begin{aligned} a_n &= (-\omega^2)^{n/2}\,a_0/(n+1)! && (n\text{ even})\,,\\ &= 0 && (n\text{ odd})\,, \end{aligned} \tag{2.28}$$

and

$$y=a_0\,x\sum_{n\ even}(-1)^{n/2}(\omega x)^n\,\frac{1}{(n+1)!}=\frac{a_0}{\omega}\,\sin\omega x\,. \tag{2.29}$$

This gives the expected solutions and so we can now start using the technique to study more complicated second order equations.

2.3 Singularities of Second Order Equations

Consider the general linear homogeneous second order equation

$$\frac{d^2y}{dx^2} + p(x)\,\frac{dy}{dx} + q(x)\,y = 0\,, \tag{2.30}$$

where $p(x)$ and $q(x)$ are functions just of x. If we want to do a power series expansion near the point x_0, the behaviour of $p(x)$ and $q(x)$ in the vicinity of this point is of crucial importance.

If $p(x)$ and $q(x)$ are finite, single valued and differentiable at x_0, then x_0 is called a *regular* point or an *ordinary* point. The equation is said to be regular at x_0. In this case $\lim_{x\to x_0} p(x)$ and $\lim_{x\to x_0} q(x)$ both exist, *i.e.* are finite. For the harmonic oscillator equation, $p(x) = 0$ and $q(x) = \omega^2$, so that all points of the equation are regular. If, on the other hand, either of the two limits is infinite, we say that x_0 is a *singular* point; the equation is singular at x_0.

Suppose that x_0 is a singular point, but that

$$\lim_{x\to 0}(x-x_0)\,p(x) \quad \text{and} \quad \lim_{x\to 0}(x-x_0)^2\,q(x) \tag{2.31}$$

both exist, the differential equation is said to have a *regular singularity* at x_0. If **either** limit is infinite, there is an *essential singularity* at $x = x_0$.

This classification is very important because if x_0 is a regular point then we can always find two independent series solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+k}\,, \tag{2.32}$$

which are convergent for all values of x between x_0 and the nearest singular point. Furthermore, it can be shown (Fuch's theorem) that if x_0 is regular singular point then there exists at least **one** series solution of the form of Eq. (2.32).

If, on the other hand, x_0 is an essential singularity of the equation then no power series solution in $x - x_0$ exists.

Roots differing by an integer

Consider the equation

$$x(x-1)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0\,.$$

Comparing this with the standard form, we see that $p(x) = 3/(x-1)$ and $q(x) = 1/x(x-1)$. Thus $x = 0$ and $x = 1$ are regular points of the differential equation and so we can expect to get at least one power series solution in x. Inserting

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+k)\, a_n\, x^{n+k-1}\,, \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, x^{n+k-2} \end{aligned}$$

into the differential equation,

$$\sum_{n=0}^{\infty} (n+k)(n+k-1)\, a_n\, (x^{n+k} - x^{n+k-1}) + \sum_{n=0}^{\infty} 3(n+k)\, a_n\, x^{n+k} + \sum_{n=0}^{\infty} a_n\, x^{n+k} = 0\,.$$

Hence

$$\sum_{n=0}^{\infty} a_n\, x^{n+k}\, [(n+k)(n+k-1) + 3(n+k) + 1] = \sum_{n=-1}^{\infty} (n+k+1)(n+k)\, a_{n+1}\, x^{n+k}\,.$$

The indicial equation comes from looking at the lowest power of x, which is given by $n = -1$ on the right hand side. This gives $k(k-1) = 0$, *i.e.* $k = 1$ or $k = 0$. The recurrence relation is

$$(n + k + 1)^2 \, a_n = (n + k + 1)(n + k) \, a_{n+1} \, ,$$

$$a_{n+1} = \left(\frac{n + k + 1}{n + k} \right) a_n \, .$$

Taking the index $k = 1$ and putting $a_0 = 1$, we get $a_1 = 2$, $a_2 = 3$ *etc.* The full solution is

$$y_1(x) = x(1 + 2x + 3x^2 + 4x^3 + \cdots) = \frac{x}{(1-x)^2} \, .$$

Note that this series converges for $| \, x \, | < 1$; the divergence at $x = 1$ is due to the singular point there.

On the other hand, when the index $k = 0$, we are in trouble because the recurrence relation is

$$a_{n+1} = \left(\frac{n + 1}{n} \right) a_n \, .$$

If you try to calculate a_1 by putting $n = 0$ you see that the whole thing blows up. Hence there is not a second *series* solution at $x = 0$. Fuch's theorem only guaranteed that there would be one solution of this kind; the other solution is going to be nasty at $x = 0$.

One can find the second (irregular) solution by letting $y(x) = y_1(x) \, v(x)$ and getting a simpler equation for $v(x)$. Normally, as in this case, $v(x)$ has a nasty $\ell n(x)$ term in it. This is always the case if the indicial equation has equal roots. This happens for Bessel's equation which one often comes across in problems with cylindrical symmetry.