3 Partial Differential Equations

3.1 Introduction

In the first year 1B21 course, you were introduced to various tricks to help in the solution of ordinary differential equations, *i.e.* ones where there is one independent and one dependent variable. Only ordinary differentiation is therefore involved. Since we live in a three-dimensional world, most differential equations are functions of the three spatial variables $(x\,,\,y\,,\,z)$ as well as perhaps the time t. One typical example is the Laplace equation

$$\nabla^2\,V(\vec{r}) = 0\,,$$

where $V(\vec{r})$ is the electrostatic potential in the region where there is no charge. The operator ∇^2, called the Laplacian, was introduced into a 1B21 problem sheet. In Cartesian coordinates

$$\nabla^2\,V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\,. \tag{3.1}$$

Another important example is the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\,\nabla^2\,\Psi(\vec{r}) + V(\vec{r})\,\Psi(\vec{r}) = E\,\Psi(\vec{r})\,, \tag{3.2}$$

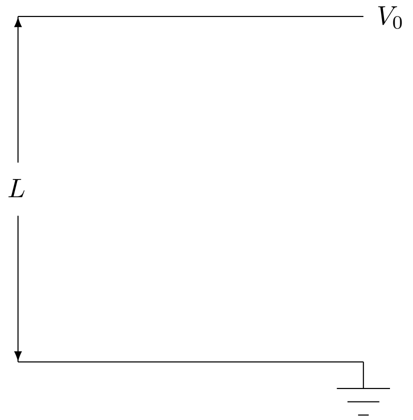
for the quantum-mechanical motion of a particle of mass m in a potential $V(\vec{r})$. $\Psi(\vec{r})$ is the particle's wave function and $\hbar = h/2\pi$, where h is Planck's constant. There are many more examples that you will come across later in your degree programme.

Superposition Principle

If V_1 and V_2 are two solutions of $\nabla^2\,V(\vec{r}) = 0$, then $V = V_1 + V_2$ is another solution. This is because the equation is linear and homogeneous. We used this principle extensively when dealing with ordinary differential equations, such as that describing simple harmonic motion, but it is equally valid for partial differential equations. This ability to add solutions is called the Superposition Principle. It is of fundamental importance in Quantum Mechanics, but I should let the 2B22 lecturer do some of the work here! We shall exploit the superposition principle extensively when trying to solve partial differential equations.

3.2 Separation in Cartesian Coordinates

Let us start with an illustrative physical example. Consider two infinitely large conducting plates. The one at $z = 0$ is earthed while that at $z = L$ is kept at a constant voltage V_0.



What is the potential between the two plates? You all know that the answer must be $V = V_0 z/L$ but we are going to derive this by solving the partial differential equation. This will demonstrate the techniques to be used in more complex cases.

Between the two plates, there is no charge and so the potential in this region satisfies Laplace's equation

$$\nabla^2 V(\vec{r}) = 0\,.$$

The boundary conditions to be applied are that, independent of the values of x and y, on the plates

$$\begin{aligned} V = 0 \quad & \text{at} \quad z = 0\,, \\ V = V_0 \quad & \text{at} \quad z = L\,. \end{aligned} \tag{3.3}$$

Since the boundary conditions are expressed easily in terms of Cartesian coordinates, it makes obvious sense to attack the problem in this coordinate system. [We could have done it in cylindrical polar coordinates, but not as an introductory problem.] In this system, Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0\,.$$

Let us try for a solution of the form

$$\begin{aligned} V(x,y,z) &= (\text{function of } x) \times (\text{function of } y) \times (\text{function of } z), \\ V(x,y,z) &= X(x)\,Y(y)\,Z(z)\,. \end{aligned} \tag{3.4}$$

At the moment we are just trying to get a single solution of the equation. If there is no solution of this kind then we will have to try something else — but of course there will be! Substituting the product form of Eq. (3.4) into Laplace's equation, we get

$$Y\,Z\,\frac{d^2X}{dx^2} + X\,Z\,\frac{d^2Y}{dy^2} + X\,Y\,\frac{d^2Z}{dz^2} = 0\,. \tag{3.5}$$

Note that we now have complete differentials (straight d's) because X is a function of only one variable (x), and similarly for Y and Z. Now divide through the equation by the product $V = X\,Y\,Z$ to get

$$\frac{1}{X}\left(\frac{d^2X}{dx^2}\right) + \frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right) + \frac{1}{Z}\left(\frac{d^2Z}{dz^2}\right) = 0\,. \tag{3.6}$$

Now the first term in Eq. (3.6) is a function only of x, the second only of y, and the third only of z. **BUT** x, y, and z are independent variables. This means that we could keep y and z fixed and vary just x. In so doing, the second and third terms remain fixed because they only depend upon y and z respectively. Hence the first term must also remain fixed even if x changes. That is, the first term is a constant, as are the second and third. Thus

$$\begin{aligned} \frac{1}{X}\left(\frac{d^2X}{dx^2}\right) &= -\ell^2\,, \\ \frac{1}{Y}\left(\frac{d^2Y}{dy^2}\right) &= -m^2\,, \\ \frac{1}{Z}\left(\frac{d^2Z}{dz^2}\right) &= +n^2\,. \end{aligned} \tag{3.7}$$

with

$$n^2 = \ell^2 + m^2\,. \tag{3.8}$$

Note that n^2, ℓ^2 and m^2 are as yet arbitrary constants and could be negative. Don't be fooled either by the choice of labels; ℓ, m, n are not necessarily integers.

We then have to solve

$$\frac{d^2X}{dx^2} = -\ell^2\,X\,. \tag{3.9}$$

For real $\ell \neq 0$, this is the simple harmonic oscillator equation which you could solve even before starting 1B21;

$$X = a_\ell \cos \ell x + b_\ell \sin \ell x\,, \tag{3.10}$$

where a_ℓ and b_ℓ are arbitrary constants which must be fixed by the boundary conditions. In the special case of $\ell = 0$, the solution simplifies to

$$X = a_0 + b_0\, x\,. \tag{3.11}$$

If you want to look what would happen if ℓ^2 were to become negative, put $\ell = il$; the $\cos \ell x$ and $sin \ell x$ become $\cosh lx$ and $i \sinh lx$. You have seen such changes before as, for example, in studying the damped oscillator in 1B27.

The solutions for Y will be similar to those for X, but with m replacing n. On the other hand, for Z we have rather

$$\left(\frac{d^2 Z}{dz^2}\right) = +n^2\, Z\,. \tag{3.12}$$

This has solutions

$$\begin{aligned} Z = & \quad e_n \cosh nz + d_n \sinh nz \qquad (n \neq 0)\,, \\ = & \qquad e_0 + f_0\, z \qquad\qquad (n = 0)\,. \end{aligned} \tag{3.13}$$

As a consequence, solutions of the separable form do exist. For example, one solution to the equation would be with $\ell = 3$, $m = 4$, and $n = 5$.

$$V(x, y, z) = \text{Constant} \times (\sin 3x) \times (\cos 4y) \times (\sinh 5z)$$

is a solution of Laplace's equation, but of course we can construct many more with different values of (ℓ, m, n).

The most general solution is of the form

$$V(x, y, z) = \text{Constant} \times \left\{ \begin{array}{c} \sin \ell x \\ \cos \ell x \end{array} \right\} \times \left\{ \begin{array}{c} \sin my \\ \cos my \end{array} \right\} \times \left\{ \begin{array}{c} \sinh nz \\ \cosh nz \end{array} \right\}$$

with the constraint that $n^2 = \ell^2 + m^2$.

By the superposition principle, any linear combination of such solutions is also a solution. The most general superposition is

$$\begin{aligned} V(x, y, z) \quad = & \quad \sum_{\ell,m} \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} \\ & \quad \times \{e_{\ell m} \cosh nz + f_{\ell m} \sinh nz\}\;. \end{aligned} \tag{3.14}$$

For any choice of ℓ and m, with $n = \sqrt{\ell^2 + m^2}$, the above product is a solution. Hence the sum is also a solution. Note that ℓ and m do not have to be integers and so the above need not be a discrete sum. Also note that if $\ell \to 0$, the cosine is replaced by 1 and the sine by x.

Imposing boundary conditions

The solution in Eq. (3.14) is quite general and we now have to relate it to the potential problem of two parallel plates. This means that we have to impose the boundary conditions.
At $z = 0$,

$$V(z = 0) = \sum_{\ell m} e_{\ell m}\, \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} = 0$$

for all values of x and y. Hence $e_{\ell m} = 0$ for all ℓ and m. The most general solution therefore simplifies to

$$V(x, y, z) = \sum_{\ell m} \sinh nz \times \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\}\;, \tag{3.15}$$

where the coefficient $f_{\ell m}$ has been absorbed into redefined $a_{\ell m}$ and $b_{\ell m}$.
At $z = L$,

$$V(z = L) = \sum_{\ell m} \sinh nL \times \{a_{\ell m} \cos \ell x + b_{\ell m} \sin \ell x\} \times \{c_{\ell m} \cos my + d_{\ell m} \sin my\} = V_0\,,$$

for all x and y. Clearly, the only solution which could lead to something which is independent of x and y is the special case of $\ell = m = n = 0$. Let us write this out explicitly as

$$V(x,y,z) = z\,\{a+bx\}\{c+dy\}\,. \tag{3.16}$$

At $z = L$,

$$V_0 = L\,\{a+bx\}\{c+dy\}$$

for all (x,y) so that $b = d = 0$ and $ac = V_0/L$. The final solution is, from Eq. (3.16), the expected

$$V = \frac{V_0 z}{L}\,.$$

Comments

1. This method of solution is called *Separation of Variables*. You look for a solution which is a product of a function of x times a function of y times a function of z. This then reduces the problem to that of solving three ordinary differential equations in x, y and z.
2. Though we have found an infinite number of solutions of the Laplace equation, we have **not** shown that we have found them all.
3. In more complicated examples the ordinary differential equations may be very much harder to solve than the simple oscillator equations here.
4. Unlike the present case, in general you cannot guess the final answer at the start!

3.3 One-dimensional Wave Equation

You should have seen in the wave equation in the first year 1B24 Waves and Optics course. In one dimension, for example a guitar string clamped at $x = 0$ and $x = L$, the displacement $y(x,t)$ obeys

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0\,, \tag{3.17}$$

where t is the time variable and c the (constant) speed of wave propagation.

Looking for a solution in the form of a product

$$y(x,t) = X(x)\,T(t) \tag{3.18}$$

leads to

$$T\,\frac{d^2X}{dx^2} - \frac{1}{c^2}\,X\,\frac{d^2T}{dt^2} = 0\,. \tag{3.19}$$

After dividing out by $y = X\,T$ and taking one term over to the right hand side, we are left with

$$\frac{1}{X}\left(\frac{d^2X}{dx^2}\right) = \frac{1}{c^2T}\left(\frac{d^2T}{dt^2}\right)\,. \tag{3.20}$$

The left hand side is a function only of x and the right hand side purely of t. Since x and t are independent variables, this means that both sides are equal to a constant, which we shall call $-\omega^2$.

The problem is therefore reduced to the solution of two ordinary differential equations

$$\begin{aligned}\left(\frac{d^2X}{dx^2}\right) + \omega^2\,X &= 0\,,\\ \left(\frac{d^2T}{dt^2}\right) + \omega^2c^2\,T &= 0\,.\end{aligned} \tag{3.21}$$

The solution of the x equation is

$$X(x) = C\cos\omega x + D\sin\omega x\,,$$

where A and B are arbitrary constants.

Since the boundary conditions are true for all time, we can impose them directly onto $X(x)$. At $x = 0$,

$$X(x=0) = 0 = C\,, \quad \Longrightarrow \quad C = 0\,,$$

whereas at $x = L$,

$$X(x=L) = 0 = D\,sin(\omega L) \quad \Longrightarrow \quad \omega = n\pi/L\,,$$

where $n = 1,\, 2,\, 3,\, \cdots$.

Solving the corresponding "t" equation,

$$\left(\frac{d^2T}{dt^2}\right) + (n\pi c/L)^2\, T = 0\,,$$

gives

$$T = A\,\cos(n\pi ct/L) + B\,\sin(n\pi ct/L)\,,$$

and a total solution of

$$y(x,t) = D\sin(n\pi x/L) \times \{A\,\cos(n\pi ct/L) + B\,\sin(n\pi ct/L)\}\ .$$

This is but one solution and, to get more, we use the superposition principle to find

$$y(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) \times \{A_n\,\cos(n\pi ct/L) + B_n\,\sin(n\pi ct/L)\}\ . \tag{3.22}$$

As before, the constant D has been absorbed into the definitions of the constants A_n and B_n.

To go further, we need to impose extra boundary conditions, such as the shape of the guitar string at time $t = 0$. We shall look at such problems later in the course when we discuss Fourier series.

3.4 Laplace's Equation in Spherical Polar Coordinates

To tie in better with the time evolution of the second year Quantum Mechanics course, we are going to switch to studying problems with spherical symmetry. If one needs to know the potential due to a charged sphere, it would be perverse to work in Cartesian coordinates. One should always choose a coordinate system which is appropriate to the boundary conditions to be imposed and, in this case, one should write things down in the spherical polar variables that were introduced in 1B21. In 1B21 the lecturer evaluated ∇^2 in plane polar coordinates and it was very messy. Unfortunately, we have to now do exactly the same in spherical polar coordinates (r,θ,ϕ), where things are even worse! We will come to a simpler derivation later in the course. Now

$$\begin{aligned} x =&\ \ r\sin\theta\cos\phi\,,\\ y =&\ \ r\sin\theta\sin\phi\,,\\ z =&\ \ r\cos\theta\,. \end{aligned} \tag{3.23}$$

The partial derivatives of the Cartesian variables with respect to the polar coordinates are

$$\begin{gathered} \frac{\partial x}{\partial r} = \sin\theta\cos\phi\,, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi\,, \quad \frac{\partial z}{\partial r} = \cos\theta\,,\\ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi\,, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi\,, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta\,.\\ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi\,, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi\,, \quad \frac{\partial z}{\partial \phi} = 0\,. \end{gathered} \tag{3.24}$$

Using the chain rule for partial differentiation, we get

$$\begin{aligned}
\frac{\partial}{\partial r} &= \sin\theta\cos\phi\,\frac{\partial}{\partial x} + \sin\theta\sin\phi\,\frac{\partial}{\partial y} + \cos\theta\,\frac{\partial}{\partial z}\ , \\
\frac{\partial}{\partial \theta} &= r\cos\theta\cos\phi\,\frac{\partial}{\partial x} + r\cos\theta\sin\phi\,\frac{\partial}{\partial y} - r\sin\theta\,\frac{\partial}{\partial z}\ , \\
\frac{\partial}{\partial \phi} &= -r\sin\theta\sin\phi\,\frac{\partial}{\partial x} + r\sin\theta\cos\phi\,\frac{\partial}{\partial y}\ .
\end{aligned} \tag{3.25}$$

These equations can be inverted to find the differentials with respect to Cartesians in terms of those with respect to polar coordinates:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\theta\cos\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\,\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\ , \\
\frac{\partial}{\partial y} &= \sin\theta\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\,\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\ , \\
\frac{\partial}{\partial z} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\ .
\end{aligned} \tag{3.26}$$

The Laplacian operator is the sum of the squares of these three operators,

$$\begin{aligned}
\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 &= \left(\sin\theta\cos\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\,\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\right)^2 \\
+ \left(\sin\theta\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\,\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial \phi}\right)^2 &+ \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\right)^2\ .
\end{aligned} \tag{3.27}$$

At this stage one has to remember that the partial derivative with respect to θ acts for example on the $\sin\theta$ as well. Taking a <u>very</u> large piece of paper, one finally ends up with

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\,\frac{\partial V}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2}\,\cot\theta\,\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 V}{\partial \phi^2}\ . \tag{3.28}$$

This is the required expression for the Laplacian operator in spherical polar coordinates.

The expression can be written in the slightly more compact form

$$\nabla^2 V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial \phi^2}\right)\ . \tag{3.29}$$

As a check on the form of the operator, consider the example of

$$V = 2x^2 - y^2 - z^2 = r^2\left(2\sin^2\theta\,\cos^2\phi - \sin^2\theta\,\sin^2\phi - \cos^2\theta\right)\ .$$

Working in Cartesian coordinates, it follows immediately that $\nabla^2 V = 0$. In spherical polar coordinates,

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) &= 6(2\sin^2\theta\,\cos^2\phi - \sin^2\theta\,\sin^2\phi - \cos^2\theta)\ . \\
\frac{\partial V}{\partial \theta} &= r^2(4\sin\theta\cos\theta\cos^2\phi - 2\sin\theta\cos\theta\sin^2\phi + 2\cos\theta\sin\theta)\ . \\
\sin\theta\frac{\partial V}{\partial \theta} &= r^2(4\sin^2\theta\cos\theta\cos^2\phi - 2\sin^2\theta\cos\theta\sin^2\phi + 2\cos\theta\sin^2\theta)\ . \\
\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial V}{\partial \theta}\right) &= 8\cos^2\theta\cos^2\phi - 4\sin^2\theta\cos^2\phi - 4\cos^2\theta\sin^2\phi \\
&\quad +2\sin^2\theta\sin^2\phi - 2\sin^2\theta + 4\cos^2\theta\ . \\
\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} &= 12\sin^2\phi - 6\ .
\end{aligned}$$

Remarkably enough, the sum of these three terms does in fact vanish!

3.5 Separation of Laplace's equation in Spherical Polar Coordinates

We want to look for a solution of the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)=0$$

in the form

$$V(r,\,\theta,\,\phi)=R(r)\times\Theta(\theta)\times\Phi(\phi)\,. \tag{3.30}$$

This involves functions which depend purely upon one variable each, *viz* r, θ and ϕ. Inserting this into Laplace's equation

$$\Theta\,\Phi\,\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+R\,\Phi\,\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+R\,\Theta\,\frac{1}{r^2\sin^2\theta}\left(\frac{d^2\Phi}{d\phi^2}\right)=0\,.$$

After dividing out by $R\,\Theta\,\Phi$ and multiplying up by $r^2\sin^2\theta$, we find

$$\frac{\sin^2\theta}{R}\,\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta}\,\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\left(\frac{d^2\Phi}{d\phi^2}\right)=0\,.$$

The first two terms here depend upon r and θ but the third is a function purely of the azimuthal angle ϕ. Since r, θ and ϕ are independent variables, this means that this third term must be some constant, which we shall denote by $-m^2$. Hence

$$\frac{\partial^2\Phi}{\partial\phi^2}=-m^2\,\Phi\,, \tag{3.31}$$

which has solutions $e^{\pm im\phi}$ or, alternatively, $\cos m\phi$ and $\sin m\phi$.

As far as the differential equation is concerned, m could have any value, even complex. However Physics imposes a fairly general boundary condition. When ϕ increases by 2π, the vector position comes back to the same point again and we should expect the same physical solution. Thus $\Phi(\phi+2\pi)=\Phi(\phi)$. This can only be accomplished if m is a real integer. Then $\Phi(\phi)$ is clearly a periodic function.

The remainder of the equation can be manipulated into the form

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{m^2}{\sin^2\theta}-\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)\,.$$

The left hand side is a function only of r, while the right hand side depends only on θ. This means that both sides must be equal to some constant that we denote by λ. This results in two ordinary differential equations;

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\lambda\,R\,, \tag{3.32}$$

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+\left(\lambda\,\sin\theta-\frac{m^2}{\sin\theta}\right)\Theta=0\,. \tag{3.33}$$

Let us now look at the radial equation of Eq. (3.32), which may be rewritten as

$$r^2\left(\frac{d^2R}{dr^2}\right)+2r\left(\frac{dR}{dr}\right)-\lambda\,R=0\,. \tag{3.34}$$

This is a special kind of homogeneous equation which is unchanged if the r-variable is scaled as $r\to\alpha\,r$, where α is some constant. We should therefore try for a solution of the form $R(r)\sim r^\beta$, since this also stays in the same form under the $r\to\alpha r$ scaling. Hence

$$\beta(\beta-1)\,r^\beta+2\beta\,r^\beta-\lambda\,r^\beta=0$$

Cancelling out the r^β factor, which cannot vanish, we see that $\beta^2+\beta=\lambda$, which has two solutions

$$\beta=\left(-1\pm\sqrt{1+4\lambda}\right)/2\,.$$

We can get exactly the same result by trying for the more general series solution. Standard manipulation leads to

$$\sum_{n=0}^{\infty} a_n \left\{(n+k)(n+k+1)-\lambda\right\} x^{n+k} = 0$$

The indicial equation, corresponding to the $n=0$ term, leads to exactly the same result as above with β replaced by k. For higher values of n we have

$$a_n \left\{(n+k)(n+k+1)-\lambda\right\} = a_n\, n(2k+1) = 0\,.$$

But $2k+1 = 2\beta+1 = \pm\sqrt{1+4\lambda}$ doesn't vanish. Hence $a_n = 0$ for $n > 1$ and we are back to the single-term solution derived above.

To make things look a bit simpler, let us define the separation constant to be $\lambda \equiv \ell(\ell+1)$, where ℓ is not necessarily an integer. Then

$$\begin{aligned} \beta &= \left(-1 \pm \sqrt{1+4\ell(\ell+1)}\right)/2 \\ &= \ell \;\text{ or }\; -\ell-1\,. \end{aligned}$$

The most general form of the radial solution is then

$$R(r) = A\,r^{\ell} + \frac{B}{r^{\ell+1}}\,. \tag{3.35}$$

In order not to interchange the two solutions, let us adopt the convention that $\ell \geq -\frac{1}{2}$.

We are then only left with the θ equation which, with the new separation constant $\ell(\ell+1)$, becomes

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1)\,\sin\theta - \frac{m^2}{\sin\theta}\right)\Theta = 0\,, \tag{3.36}$$

which does not look very attractive. It looks a little more tractable if we use the variable $\mu = \cos\theta$ rather than θ. Then $d\mu/d\theta = -\sin\theta$ and

$$\frac{d}{d\theta} = -\sin\theta\,\frac{d}{d\mu} = -\sqrt{1-\mu^2}\,\frac{d}{d\mu}\,.$$

Hence

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]\Theta = 0\,. \tag{3.37}$$

This is the famous Legendre differential equation which the 2B22 Quantum Mechanics lecturer should be reaching soon. Legendre discovered his equation when trying to interpret planetary gravitational fields, "Recherches sur la figure des planètes" (1784). This is about 150 years before the discovery of the Schrödinger equation and so you shouldn't blame quantum mechanics for the introduction of Legendre polynomials.