6 Vector Operators

6.1 The Gradient Operator

In the 1B21 course you were introduced to the gradient operator in Cartesian coordinates. For any differentiable scalar function $f(x, y, z)$, we can define a vector function through

$$grad\, f = \underline{\nabla}\, f = (f_x\underline{\hat{e}}_x + f_y\underline{\hat{e}}_y + f_z\underline{\hat{e}}_z) = \left(\frac{\partial f}{\partial x}\right)\underline{\hat{e}}_x + \left(\frac{\partial f}{\partial y}\right)\underline{\hat{e}}_y + \left(\frac{\partial f}{\partial z}\right)\underline{\hat{e}}_z\,. \tag{6.1}$$

There is a slight difference with the notation of the first year course since $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ have been replaced by $\underline{\hat{e}}_x$, $\underline{\hat{e}}_y$ and $\underline{\hat{e}}_z$ to allow a more straightforward generalisation to polar coordinates.

Sometimes we express the result in terms of an operator equation

$$\underline{\nabla} = \underline{\hat{e}}_x\,\frac{\partial}{\partial x} + \underline{\hat{e}}_y\,\frac{\partial}{\partial y} + \underline{\hat{e}}_z\,\frac{\partial}{\partial z}\,, \tag{6.2}$$

where it is understood that the operator $\underline{\nabla}$, pronounced **Del**, is to act upon the function f.

If the point (x, y, z) is changed by an infinitesimal amount $\underline{dr} = (dx, dy, dz)$, the function changes by

$$df = (f_x\underline{\hat{e}}_x + f_y\underline{\hat{e}}_y + f_z\underline{\hat{e}}_z)\cdot(dx\,\underline{\hat{e}}_x + dy\,\underline{\hat{e}}_y + dz, \underline{\hat{e}}_z) = f_x\,dx + f_y\,dy + f_z\,dz\,, \tag{6.3}$$

which is just the expression for the total derivative in terms of the slopes in the three directions.

Upon the surface $f(x, y, z) =$ constant, $df = 0$, so that

$$0 = \underline{\nabla}\, f\,\cdot\,\underline{dr}\,. \tag{6.4}$$

In order that the point stays along the contour, small changes $\underline{dr}$ are perpendicular to the gradient.

Examples

1. Find the gradient vectors $\underline{\nabla}\,\phi$ and $\underline{\nabla}\,\psi$ of the functions

$$\begin{aligned}\phi &= x^2 + y^2 - z^2 + 3\,,\\ \psi &= xy - yz + zx - 10\,,\end{aligned}$$

at the point $(3, 2, 4)$, and the acute angle between these two directions correct to $0.1°$.

The two gradients are

$$\begin{aligned}\underline{\nabla}\,\phi &= (2x,\, 2y,\, -2z)\,,\\ \underline{\nabla}\,\psi &= (y + z,\, x - z,\, x - y)\,.\end{aligned}$$

At the point $(3, 2, 4)$,

$$\underline{\nabla}\,\phi = (6,\, 4,\, -8)\,, \qquad \underline{\nabla}\,\psi = (6,\, -1,\, 1)\,.$$

If θ is the angle between these two vectors, then

$$\cos\theta = \frac{(6,\, 4,\, -8)\cdot(6,\, -1,\, 1)}{2\sqrt{29\times 38}} = \frac{12}{\sqrt{1102}}\,.$$

Hence the acute angle $\theta \approx 68.8^0$.

2. In the case where f is only a function of $r = \sqrt{x^2 + y^2 + z^2}$, then

$$\left(\frac{\partial f}{\partial x}\right) = \frac{df}{dr}\left(\frac{\partial\sqrt{x^2 + y^2 + z^2}}{\partial x}\right) = \frac{x}{r}\,\frac{df}{dr}\,,$$

where we have used a chain rule for differentiating. Thus

$$grad\, f = (x,\, y,\, z)\,\frac{1}{r}\,\frac{df}{dr} = \underline{\hat{e}}_r\,\frac{df}{dr}\,,$$

where $\underline{\hat{e}}_r$ is a unit vector pointing in the direction of $\underline{r}$.

6.2 The Divergence Operator

In the previous section, we used the $\underline{\nabla}$ (Del) operator to produce a vector field *grad f* from a scalar field f. The divergence operator does the opposite — it creates a scalar field from a vector using the scalar product. For a differentiable vector function $\underline{v}(x, y, z)$,

$$div\,\underline{v} = \underline{\nabla} \cdot \underline{v} = \left(\underline{\hat{e}}_x \frac{\partial}{\partial x} + \underline{\hat{e}}_y \frac{\partial}{\partial y} + \underline{\hat{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(v_x \underline{\hat{e}}_x + v_y \underline{\hat{e}}_y + v_z \underline{\hat{e}}_z\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\,. \tag{6.5}$$

Examples

1. In the case where $\underline{v} = \underline{r}$,

$$div\,\underline{r} = \underline{\nabla} \cdot \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3\,.$$

2. If f is a function of the magnitude of $\underline{r}$,

$$div\,[\underline{r}\,f(r)] = \frac{\partial[x\,f(r)]}{\partial x} + \frac{\partial[y\,f(r)]}{\partial y} + \frac{\partial[z\,f(r)]}{\partial z} = f + \frac{x^2}{r}\frac{df}{dr} + f + \frac{y^2}{r}\frac{df}{dr} + f + \frac{z^2}{r}\frac{df}{dr}$$

$$= 3f(r) + r\,\frac{df}{dr}\,.$$

3. Suppose that $f(x, y, z)$ and $\underline{v}(x, y, z)$ are respectively scalar and vector functions of the coordinates (x, y, z). Then

$$\underline{\nabla} \cdot (f\,\underline{v}) = \frac{\partial}{\partial x}(f\,v_x) + \frac{\partial}{\partial y}(f\,v_y) + \frac{\partial}{\partial z}(f\,v_z)$$

$$= v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} + f\,\frac{\partial v_x}{\partial x} + f\,\frac{\partial v_y}{\partial y} + f\,\frac{\partial v_z}{\partial z} = \underline{v} \cdot \underline{\nabla} f + f\,\underline{\nabla} \cdot \underline{v}\,.$$

   The vector part of the operator obeys the rules for vectors, whereas the differentiation part works obeys the normal rules for differentiation, including that for the derivative of a product.

6.3 The Curl Operator

We saw in the 1B21 course that the differential

$$dW = v_x\,dx + v_y\,dy + v_z\,dz\,, \tag{6.6}$$

is perfect if and only if

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = \frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} = \frac{\partial v_z}{\partial x} - \frac{\partial v_y}{\partial z} = 0\,. \tag{6.7}$$

Let us introduce the more compact notation using the *curl* of a vector function $\underline{v} = (v_x,\, v_y,\, v_z)$, defined as

$$curl\,\underline{v} = \underline{\nabla} \times \underline{v}\,. \tag{6.8}$$

Using the expression for the cross product given in the 1B21 course, together with the expression for Del given by Eq. (6.2), it is seen that this has components

$$\begin{aligned}
(\underline{\nabla} \times \underline{v})_x &= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\,,\\
(\underline{\nabla} \times \underline{v})_y &= \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\,,\\
(\underline{\nabla} \times \underline{v})_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\,.
\end{aligned} \tag{6.9}$$

Thus, if $curl\,\underline{v} = 0$, then the integral

$$W(x, y, z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \underline{v} \cdot \underline{dr} \tag{6.10}$$

defines a *unique* function $W(x, y, z)$, whose value does not depend upon the path of integration.
Examples

1. We saw, in the discussion of the divergence of a product, that $\underline{\nabla}$ obeys simultaneously the rules of differentiation and vector algebra. This is also the case for *curl*. Thus

$$\underline{\nabla} \times (f\,\underline{v}) = f\,(\underline{\nabla} \times \underline{v}) + (\underline{\nabla}\,f) \times \underline{v}\,,$$

where the convention is that the differentiation on the right hand side only takes place inside the bracket. The simplest proof is in terms of components. Taking just the x-component of the LHS,

$$(\mathrm{LHS})_x = \frac{\partial}{\partial y}(f\,v_z) - \frac{\partial}{\partial z}(f\,v_y)$$

$$= \left(\frac{\partial f}{\partial y}\,v_z - \frac{\partial f}{\partial z}\,v_y\right) + f\,\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) = [(\underline{\nabla}\,f) \times \underline{v}]_x + f\;[\underline{\nabla} \times \underline{v}]_x\;,$$

and similarly for the other components.

2. If now $\underline{v} = \underline{r}$ and $f = f(r)$, what is $\underline{\nabla} \times (\underline{r}\,f(r))$?

$$\underline{\nabla} \times \underline{r} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \\ \underline{\hat{e}}_x & \underline{\hat{e}}_y & \underline{\hat{e}}_z \end{vmatrix} = 0\,,$$

since the x partial differentiation acts here on y and z, but not x.

We have already shown that

$$\underline{\nabla}\,f(r) = \underline{\hat{e}}_r\,\frac{df}{dr} = \frac{\underline{r}}{r}\,\frac{df}{dr}\,.$$

Since

$$\underline{r} \times \left(\frac{\underline{r}}{r}\,\frac{df}{dr}\right) = 0\,,$$

this means that

$$\underline{\nabla} \times (\underline{r}\,f(r)) = 0\,.$$

6.4 Operators quadratic in $\underline{\nabla}$

The gradient operator takes a scalar into a vector. Acting on the result with the divergence operator gives a scalar again.

$$div\,grad\,f = \underline{\nabla} \cdot (\underline{\nabla}\,f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,. \tag{6.11}$$

The resulting operator ∇^2 is called the Laplacian operator and it has already been used when discussing the Legendre polynomials. Most of Physics seems to be governed by second order differential equations involving the Laplacian operator. In the Quantum Mechanics course you have been looking at the Schrödinger equation describing the motion of a particle of mass m with energy E in a potential $V(r)$;

$$-\frac{\hbar^2}{2m}\nabla^2\,\Psi + V(r)\,\Psi = E\,\Psi\,. \tag{6.12}$$

In electrostatics, you learned that the potential $\Phi(\underline{r})$ due to a charge density $\rho(\underline{r})$ satisfies the equation

$$\nabla^2\Phi = -\frac{1}{4\pi\varepsilon_0}\,\rho\,. \tag{6.13}$$

There are, however, many more examples.

There are other operators which are quadratic in $\underline{\nabla}$, *e.g.*

$$\underline{A} = curl\, grad\, \phi = \underline{\nabla} \times (\underline{\nabla}\, \phi) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ \underline{\hat{e}}_x & \underline{\hat{e}}_y & \underline{\hat{e}}_z \end{vmatrix} . \tag{6.14}$$

This has an x-component of

$$A_x = \frac{\partial}{\partial y}\frac{\partial}{\partial z}\,\phi - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\,\phi = 0 \tag{6.15}$$

for any reasonable function $\phi(x, y, z)$. Thus

$$\underline{\nabla} \times (\underline{\nabla}\,\phi) = \underline{0}\,. \tag{6.16}$$

You should all recognise this result in the case of an electrostatic field with $\underline{E} = -\underline{\nabla}\,\phi$. The electrostatic field is irrotational

$$\underline{\nabla} \times \underline{E} = \underline{0}\,. \tag{6.17}$$

By writing things out in component form, one can easily show that

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0\,. \tag{6.18}$$

This is another useful result in electromagnetism. The magnetic induction field $\underline{B}$ is *solenoidal*, *i.e.* $\underline{\nabla} \cdot \underline{B} = 0$. Hence, using Eq. (6.18), we can write

$$\underline{B} = \underline{\nabla} \times \underline{A}\,. \tag{6.19}$$

In Electromagnetism, $\underline{A}$ is called the magnetic vector potential. Though this is not currently used in the second year E&M course, it will be needed later in the quantum description of the interaction of radiation with matter. The magnetic potential seems just to be some artificial construct introduced to make $\underline{B}$ automatically solenoidal and so it needn't correspond directly to a measurable physical quantity. Nevertheless, the Aharanov-Bohm effect in Quantum Mechanics shows that certain features of the magnetic potential have experimental consequences!

Another quadratic relation is

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A}\,. \tag{6.20}$$

In words this is

curl (curl $\underline{A}$) = grad (div $\underline{A}$) – del squared $\underline{A}$.

This can be proved by writing everything explicitly in terms of components, but there are other methods. In the 1B21 course, it was shown that for ordinary vectors

$$\underline{d} = \underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\,\underline{b} - (\underline{a} \cdot \underline{b})\,\underline{c}\,. \tag{6.21}$$

The trouble about using this relation is that, when $\underline{b}$ is also a differential operator, we are not allowed to change the order at will. Hence write it in the symbolic component form

$$d_i = \sum_j (a_j\, b_i\, c_j - a_j\, b_j\, c_i)\,, \tag{6.22}$$

where we have *NOT* altered the order of the vectors. Now put $\underline{a} = \underline{b} = \underline{\nabla}$ and $\underline{c} = \underline{A}$. Since $\underline{a}$ and $\underline{b}$ are now the same operators, it doesn't matter in which order we write them. Hence

$$d_i = \sum_j (\nabla_i \nabla_j A_j - \nabla_j \nabla_j A_i)\,, \tag{6.23}$$

which is just the component representation of Eq. (6.20).

6.5 The Divergence Theorem

Two-dimensional integrals were discussed in the 1B21 course. These have to be generalised a bit in order to introduce the *Flux of a Vector Field*.

An infinitesimal integration element $dS = dx\,dy$ is a small bit of the $x - y$ plane of area $dx \times dy$. Planes, however, have directions given by their normals. In this case, the normal is along the z-axis, which means that the integration element is really the vector

$$\underline{dS} = dx\,dy\,\underline{\hat{e}}_z\ . \tag{6.24}$$

This is just one particular example, but in general

$$\underline{dS} = dS\,\hat{n}\ , \tag{6.25}$$

where $\hat{n}$ is the normal to this small bit of plane.

To understand the concept of vector flux, consider a simple example from fluid flow where a liquid of density ρ is moving with velocity $\underline{v}$ through a surface S. The amount of mass that passes through an element $\underline{dS}$ per unit time depends upon the component of $\underline{v}$ perpendicular to $\underline{dS}$;

$$dm = \rho\,v\cos\theta\,dS = \rho\,\underline{v}\cdot\underline{dS}\ , \tag{6.26}$$

where θ is the angle between $\underline{v}$ and $\hat{n}$.

If we define a vector field by $\underline{F} = \rho\,\underline{v}$, then the rate of flow (flux) of mass per unit time is

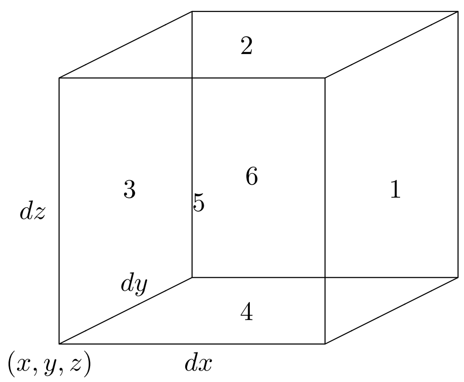
$$dm = \underline{F}\cdot\underline{dS}\ , \tag{6.27}$$

and the total flux through the surface is

$$m = \int_S \underline{F}\cdot\underline{dS} = \int_S \underline{F}\cdot\hat{n}\,dS\ . \tag{6.28}$$

You will have seen integrals of this type in Faraday's law of Magnetic Induction.

Consider now the flux of a vector through a closed surface, such as the infinitesimal cube in the picture.



We are going to build up the total flux through the six sides by evaluating each face separately, starting with face-1. The normal to this surface points in the positive x-direction, so that $\underline{dS} = dy\,dz\,\underline{\hat{e}}_x$. Note however that the value of the x-coordinate is $x + dx$, since the lower right-hand corner has been taken to be (x, y, z). Therefore the flux is

$$I_1 = \int \underline{F}(x+dx, y, z)\cdot\underline{\hat{e}}_x\,dy\,dz = F_x(x+dx, y, z)\,dy\,dz\ . \tag{6.29}$$

On face-2 the normal to the surface points in the *negative* x-direction, so that $\underline{dS} = -dy\,dz\,\underline{\hat{e}}_x$ and $I_2 = F_x(x, y, z)\,dy\,dz$. The sum of the flux through these two faces is

$$I_1 + I_2 = \left(F_x(x+dx, y, z) - F_x(x, y, z)\right)\,dy\,dz\ . \tag{6.30}$$

Since dx is very small, we can expand F_x in a Taylor series in dx, keeping just the first two terms:

$$F_x(x + dx, y, z) \approx F_x(x, y, z) + dx\, \frac{\partial}{\partial x} F_x(x, y, z)\,. \tag{6.31}$$

$$I_1 + I_2 = \frac{\partial}{\partial x} F_x(x, y, z)\, dx\, dy\, dz\,. \tag{6.32}$$

Adding the contributions from the other two pairs of sides,

$$\int_S \underline{F} \cdot \underline{dS} = (\underline{\nabla} \cdot \underline{F})\, dx\, dy\, dz = (\underline{\nabla} \cdot \underline{F})\, dV\,. \tag{6.33}$$

This is the infinitesimal form of the divergence theorem. The integral over a closed surface of the flux of a vector field is equal to the volume integral of the divergence of the vector.

$$\int_V (\underline{\nabla} \cdot \underline{F})\, dV = \int_S \underline{F} \cdot \underline{dS}\,. \tag{6.34}$$

If we put two such cubes together, the flux terms cancel on the common surface because the normals are in opposite directions. The volume terms just add. Hence the integral form of Eq. (6.34) is valid for the two cubes together. We can build up any closed shape if we take enough infinitesimal cubes. Try building a model of Big Ben out of Lego bricks! The divergence theorem is therefore true generally.

The divergence theorem is much used in electrostatics, where the divergence of the displacement vector $\underline{D}$ is proportional to the charge density;

$$\underline{\nabla} \cdot \underline{D} = \frac{1}{\varepsilon_0} \rho(\underline{r})\,. \tag{6.35}$$

Integrating this over a volume V and using the divergence theorem, we have

$$\frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho\, dV = \int_V (\underline{\nabla} \cdot \underline{D})\, dV = \int_S \underline{D} \cdot \underline{dS}\,. \tag{6.36}$$

The electric flux through a closed surface S is equal to the amount of charge contained therein. This is known as Gauss's theorem.

As a simple application of the law, consider a spherically symmetric charge distribution. This gives rise to a spherically symmetric electric field $\underline{D} = D_r\, \underline{\hat{e}}_r$. If we take the surface S to be that of a sphere of radius r, then the normal to the surface lies along the radius vector. Hence

$$\int_S \underline{D} \cdot \underline{dS} = 4\pi r^2\, D_r = \frac{Q}{\varepsilon_0}\,. \tag{6.37}$$

This gives the standard result that $D_r = Q/4\pi\varepsilon_0 r^2$.

Example

Find the divergence $\underline{\nabla} \cdot \underline{A}$ of the field

$$\underline{A} = xy\underline{\hat{e}}_x + y\underline{\hat{e}}_y + z^3\underline{\hat{e}}_z\,.$$

For a right cylinder defined by $x^2 + y^2 \le 1$ and $-1 \le z \le +1$, derive the volume integral of the divergence of $\underline{A}$. Verify the divergence theorem in this case.

$$\underline{\nabla} \cdot \underline{A} = \frac{\partial(xy)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z^3)}{\partial z} = y + 1 + 3z^2\,.$$

Integrating over the cylinder,

$$I = \int_V \underline{\nabla} \cdot \underline{A}\, dV = \int_0^1 r\, dr \int_0^{2\pi} d\theta \int_{-1}^{+1} dz\, (y + 1 + 3z^2) = \int_0^1 r\, dr \int_0^{2\pi} d\theta\, \left[(y+1)z + z^3\right]_{-1}^{+1}\,.$$

To proceed further, change to polar coordinates $y = r\sin\theta$. Since the integral of $\sin\theta$ from 0 to 2π then vanishes, we are left with

$$I = \int_0^1 r\,dr(8\pi) = 4\pi\,.$$

To verify the divergence theorem, split the surface integral into ones over the caps of the cylinder and one over the curved part.
On the top cap, $\hat{n} = \underline{\hat{e}}_z$ and $\underline{A}\cdot\hat{n} = z^3 = 1$.

$$I_1 = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \pi\,.$$

On the bottom cap, the outward normal changes sign, but so does z^3, which means that this surface integral is also $I_2 = \pi$.

On the curved surface $\hat{n} = \hat{r} = \cos\theta\,\underline{\hat{e}}_x + \sin\theta\,\underline{\hat{e}}_y$, so that

$$\underline{A}\cdot\hat{n} = xy\cos\theta + y\sin\theta = \cos^2\theta\sin\theta + \sin^2\theta\,.$$

$$I_3 = \int_0^{2\pi} d\theta \int_{-1}^{+1} dz\left(\cos^2\theta\sin\theta + \sin^2\theta\right) = 2\int_0^{2\pi} d\theta\,\left(\cos^2\theta\sin\theta + \sin^2\theta\right)\,.$$

The first integral on the RHS vanishes because

$$\int_0^{2\pi} d\theta\cos^2\theta\sin\theta = -\int_{\theta=0}^{\theta=2\pi} \cos^2\theta\,d(\cos\theta) = \cos^3\theta\Big|_{\theta=0}^{\theta=2\pi} = 0\,.$$

On the other hand,

$$\int_0^{2\pi} d\theta\sin^2\theta = \tfrac{1}{2}\int_0^{2\pi} d\theta\,(1-\cos 2\theta) = \tfrac{1}{2}[\theta - \sin 2\theta/2]_0^{2\pi} = \pi\,.$$

Hence $I_3 = 2\pi$ and $I_1 + I_2 + I_3 = 4\pi$, which verifies the divergence theorem in this case.

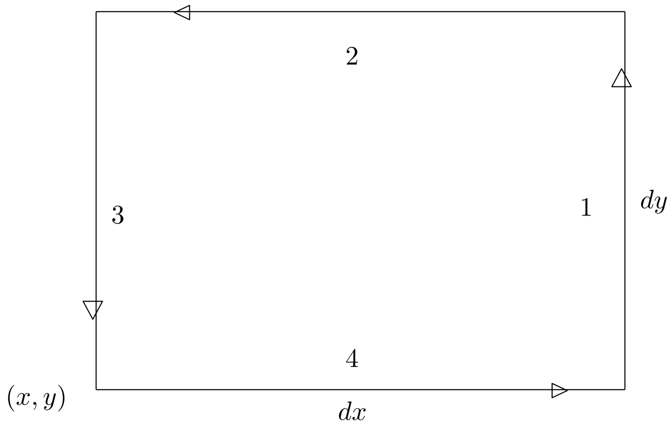
6.6 Stokes' Theorem

Stokes' theorem states that the surface integral of the curl of a vector is equal to the line integral of the vector along the contour surrounding this area

$$\int_S (\underline{\nabla}\times\underline{F})\cdot\underline{dS} = \int_L \underline{F}\cdot\underline{d\ell}\,. \tag{6.38}$$

We have to be slightly careful here. The sign of the normal to the surface is given by the corkscrew (right-hand) rule when going round the contour L.

The method of proof is very similar to that of the divergence theorem. We start by taking an infinitesimal rectangle in the $x-y$ plane, as in the picture.



Start at the bottom-right corner and go round anti-clockwise, 1, 2, 3, and 4. Since, along path-1, $\underline{d\ell}$ is parallel to $\underline{\hat{e}}_y$,

$$I_1 = \int_y^{y+dy} F_y(x+dx, y)\, dy \approx F_y(x+dx, y)\, dy \tag{6.39}$$

for small dy. On the other hand, along path-3, $\underline{d\ell}$ points in the *negative* y-direction, so that $I_3 \approx -F(x,y)\, dy$. The sum of these two contributions is

$$I_1 + I_3 = (F_y(x+dx, y) - F_y(x,y))\; dy \approx \frac{\partial}{\partial x} F_y(x,y)\, dx\, dy\, . \tag{6.40}$$

Adding the contributions from the other two paths, we see that

$$\int_L \underline{F} \cdot \underline{d\ell} = \left(\frac{\partial}{\partial x} F_y(x,y) - \frac{\partial}{\partial y} F_x(x,y) \right) dx\, dy = (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}\, , \tag{6.41}$$

since $\underline{dS}$ lies in the positive z-direction (using the right-hand rule convention).

This proves Stokes' theorem for the very small rectangle. Any surface can be constructed as a sum of infinitesimal plane rectangles so that we can sew them together and prove the theorem for an area of arbitrary shape in three dimensions. Note that along the common line between two such rectangles, the line integrals cancel, so that only the path around the periphery of the combined area is left.

Example

Verify Stokes' theorem for the vector field $\underline{A} = 2y\, \underline{\hat{e}}_x + 3x\, \underline{\hat{e}}_y - z^2\, \underline{\hat{e}}_z$ and a hemispherical surface $x^2+y^2+z^2 = 9$ for which $z \geq 0$.

Start by working out $curl\, \underline{A}$.

$$\underline{\nabla} \times \underline{A} = \begin{vmatrix} \underline{\hat{e}}_x & \underline{\hat{e}}_y & \underline{\hat{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = [0-0]\underline{\hat{e}}_x - [0-0]\underline{\hat{e}}_y + [3-2]\underline{\hat{e}}_z = \underline{\hat{e}}_z\, .$$

Since the surface is that of a hemisphere centred at the origin, the normal to the surface is

$$\hat{n} = \hat{r} = \frac{x\underline{\hat{e}}_x + y\underline{\hat{e}}_y + z\underline{\hat{e}}_z}{r}\, ,$$

and

$$(\underline{\nabla} \times \underline{A}) \cdot \hat{n} = \frac{z}{r} = \cos\theta\, .$$

It is easiest to do the integration over the surface in polar coordinates, where $dS = 9\sin\theta\, d\theta\, d\phi$;

$$\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 9 \int_0^{\pi} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\phi = 9\pi \int_0^{\pi} \sin 2\theta\, d\theta \ = \frac{9\pi}{2}[\cos 2\theta]_0^{\pi} = 9\pi\ .$$

On the other hand, along the circle $x^2 + y^2 = 9$, we use plane polar coordinates $x = 3\cos\phi$, $y = 3\sin\phi$, $\underline{d\ell} = (-3\sin\phi\, \underline{\hat{e}}_x + 3\cos\phi\, \underline{\hat{e}}_y)d\phi$.

$$\int_L \underline{A} \cdot \underline{d\ell} = \int_0^{2\pi} (6\sin\phi\, \underline{\hat{e}}_x + 9\cos\phi\, \underline{\hat{e}}_y) \cdot (-3\sin\phi\, \underline{\hat{e}}_x + 3\cos\phi\, \underline{\hat{e}}_y)\, d\phi$$

$$= \int_0^{2\pi} (27\cos^2\phi - 18\sin^2\phi)\, d\phi = 2\pi\left[\frac{27}{2} - 9\right] = 9\pi\ .$$

The last integral can be done with the standard trigonometric identities, but I used the result that, averaged over one period, $< \cos^2\phi >=< \sin^2\phi >= \frac{1}{2}$.

6.7 Coordinate-Independent Definitions

The forms of *div, grad* and *curl* have been defined in Cartesian coordinates, but we must now evaluate them in other coordinate systems. The three formulae needed are

$$df = (\underline{\nabla} f) \cdot \underline{dr}\ , \tag{6.42}$$

for any infinitesimal change in the position vector $\underline{r}$.

$$\underline{\nabla} \cdot \underline{F} = \lim_{V\to 0}\left\{\frac{1}{V}\int_S \underline{F} \cdot \underline{dS}\right\} \tag{6.43}$$

for a very small volume.

$$(\underline{\nabla} \times \underline{F}) \cdot \hat{S} = \lim_{S\to 0}\left\{\frac{1}{S}\int_L \underline{F} \cdot \underline{d\ell}\right\} \tag{6.44}$$

for the component of *curl* in the $\hat{S}$ direction.

6.8 Spherical Polar Coordinates

It would be ridiculous to work out the potential due to a charged sphere in Cartesian coordinates rather than spherical polars. Similarly, for a dielectric cylinder, use cylindrical polar coordinates. Three-dimensional problems are quite difficult enough; the symmetry of the problem must be used to simplify the problems as much as possible. In mathematics books like Arfken and Weber, you will find that there are 14 *useful* coordinate systems! However, in all my research areas, I have only ever used four and in this course I am only going to discuss three, *viz* Cartesian, Spherical Polar, and Cylindrical Polar coordinates.

The Cartesian components of a point $\vec{r}$ in spherical polar coordinates (r, θ, ϕ) are given by

$$\begin{aligned} x =&\ r\sin\theta\cos\phi\ , \\ y =&\ r\sin\theta\sin\phi\ , \\ z =&\ r\cos\theta\ . \end{aligned} \tag{6.45}$$

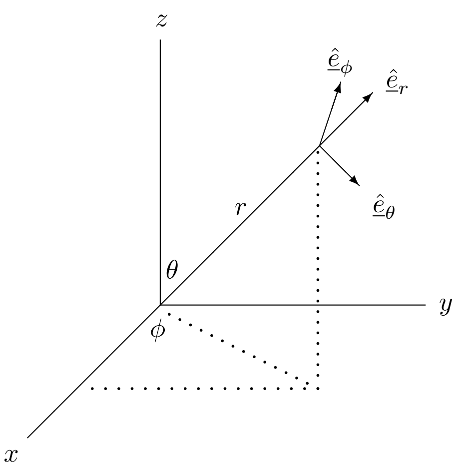
The basis vectors in this system can be found geometrically if you are good at drawing in three dimensions. They are

$$\begin{aligned} \underline{\hat{e}}_r =&\ \sin\theta\cos\phi\, \underline{\hat{e}}_x + \sin\theta\sin\phi\, \underline{\hat{e}}_y + \cos\theta\, \underline{\hat{e}}_z\ , \\ \underline{\hat{e}}_\theta =&\ \cos\theta\cos\phi\, \underline{\hat{e}}_x + \cos\theta\sin\phi\, \underline{\hat{e}}_y - \sin\theta\, \underline{\hat{e}}_z\ , \\ \underline{\hat{e}}_\phi =&\ -\sin\phi\, \underline{\hat{e}}_x + \cos\phi\, \underline{\hat{e}}_y\ , \end{aligned} \tag{6.46}$$

which satisfy

$$\hat{\underline{e}}_r \times \hat{\underline{e}}_\theta = \hat{\underline{e}}_\phi. \tag{6.47}$$

This shows that the basis vectors in this system form a right-handed perpendicular system.



Differentiating Eq. (6.45) with respect to r, θ, and ϕ, we see that

$$\underline{dr} = dr\, \hat{\underline{e}}_r + r\, d\theta\, \hat{\underline{e}}_\theta + r \sin\theta\, d\phi\, \hat{\underline{e}}_\phi\, . \tag{6.48}$$

This can also be seen geometrically with a nice picture.

The volume element is the product of these three terms (much simpler than working out the Jacobian)

$$dV = r^2\, \sin\theta\, dr\, d\theta\, d\phi\, , \tag{6.49}$$

and the elements of area pointing in the directions of the basis vectors

$$\begin{aligned} dS_r &= r^2\, \sin\theta\, d\theta\, d\phi\, , \\ dS_\theta &= r \sin\theta\, dr\, d\phi\, , \\ dS_\phi &= r\, dr\, d\theta\, . \end{aligned} \tag{6.50}$$

Boas, and other books, evaluate everything for a general coordinate system in terms of the so-called scale-factors of the metric h_i, defined by

$$\underline{dr} = \sum_{i=1}^{3} h_i\, (dx)_i\, \hat{\underline{e}}_i\, . \tag{6.51}$$

Since we are here only going to look at two coordinate systems, I prefer to do things explicitly and *not* use the h_i.

We first work out the expression for the gradient in spherical polar coordinates:

$$\underline{\nabla}\, f = (\underline{\nabla}\, f)_r\, \hat{\underline{e}}_r + (\underline{\nabla}\, f)_\theta\, \hat{\underline{e}}_\theta + (\underline{\nabla}\, f)_\phi\, \hat{\underline{e}}_\phi\, . \tag{6.52}$$

From Eq. (6.42),

$$\begin{aligned} df = (\underline{\nabla}\, f) \cdot \underline{dr} &= ((\underline{\nabla}\, f)_r\, \hat{\underline{e}}_r + (\underline{\nabla}\, f)_\theta\, \hat{\underline{e}}_\theta + (\underline{\nabla}\, f)_\phi\, \hat{\underline{e}}_\phi) \cdot (dr\, \hat{\underline{e}}_r + r\, d\theta\, \hat{\underline{e}}_\theta + r \sin\theta\, d\phi\, \hat{\underline{e}}_\phi) \\ &= (\underline{\nabla}\, f)_r\, dr + (\underline{\nabla}\, f)_\theta\, r\, d\theta + (\underline{\nabla}\, f)_\phi\, r \sin\theta\, d\phi\, . \end{aligned} \tag{6.53}$$

But, from the chain rule for partial differentiation,

$$df = \frac{\partial f}{\partial r}\, dr + \frac{\partial f}{\partial \theta}\, d\theta + \frac{\partial f}{\partial \phi}\, d\phi\, . \tag{6.54}$$

Comparing the Eqs. (6.53) and (6.54), we can read off the spherical polar components of the gradient;

$$\begin{aligned} (\underline{\nabla}\, f)_r &= \frac{\partial f}{\partial r}\,, \\ (\underline{\nabla}\, f)_\theta &= \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,, \\ (\underline{\nabla}\, f)_\phi &= \frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \phi}\,. \end{aligned} \tag{6.55}$$

To find the expression for the divergence, evaluate the flux of the vector $\underline{F}(r,\theta\,\phi)$ through the sides of the little hypercube $(dr\,, r\,d\theta\,, r\sin\theta\,d\phi)$. The net flux through the two sides with constant r are

$$\begin{aligned} I_1 + I_2 &= F_r(r+dr,\theta,\phi)\,(r+dr)^2\,\sin\theta\,d\theta\,d\phi - F_r(r,\theta,\phi)\,r^2\,\sin\theta\,d\theta\,d\phi \\ &\approx \frac{\partial}{\partial r}\left(r^2\,F_r(r,\theta,\phi)\right)\,dr\,\sin\theta\,d\theta\,d\phi\,. \end{aligned} \tag{6.56}$$

Note that, not only does F_r change when r increases, but so does the area factor r^2.

Doing the same thing for the change in the θ and ϕ coordinates,

$$\begin{aligned} I_3 + I_4 &= F_\theta(r,\theta+d\theta,\phi)\,r\,\sin(\theta+d\theta)\,dr\,d\phi - F_\theta(r,\theta,\phi)\,r\,\sin\theta\,dr\,d\phi \\ &\approx \frac{\partial}{\partial \theta}\,(\sin\theta\,F_\theta(r,\theta,\phi))\,r\,dr\,d\theta\,d\phi\,. \end{aligned} \tag{6.57}$$

$$I_5 + I_6 = F_\phi(r,\theta,\phi+d\phi)\,r\,dr\,d\theta - F_\phi(r,\theta,\phi)\,r\,dr\,d\theta \approx \frac{\partial}{\partial \phi}\,(F_\phi(r,\theta,\phi))\,r\,dr\,d\theta\,d\phi\,. \tag{6.58}$$

Thus, from Eq. (6.43),

$$\begin{aligned} \underline{\nabla}\cdot\underline{F} &= \frac{1}{r^2\sin\theta\,dr\,d\theta\,d\phi}\times \\ &\left\{\frac{\partial}{\partial r}\left(r^2 F_r(r,\theta,\phi)\right)dr\sin\theta\,d\theta\,d\phi + \frac{\partial}{\partial\theta}(\sin\theta\,F_\theta(r,\theta,\phi))r\,dr\,d\theta\,d\phi + \frac{\partial}{\partial\phi}(F_\phi(r,\theta,\phi))r\,dr\,d\theta\,d\phi\right\} \\ &= \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,F_r(r,\theta,\phi)\right) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,F_\theta(r,\theta,\phi)) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\phi}\,(F_\phi(r,\theta,\phi))\,. \end{aligned} \tag{6.59}$$

Note that, even if all of the components of $\underline{F}$ are constant, the divergence does not vanish because of the changing geometry in spherical polar coordinates.

Since we now have expressions for both the gradient and divergence in spherical polar coordinates, we can also work out that for the Laplacian by substituting Eq. (6.55) into Eq. (6.59);

$$\nabla^2 V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\phi}\right)\,. \tag{6.60}$$

This agrees with the formula of Eq. (2.29) that we obtained by manipulating the partial differentiatals.

Using exactly the same techniques on Stokes' theorem allows us to obtain the expression for the curl. To get the r-component, look at the line integral around the elementary contour of the figure.

$r\sin(\theta)\,d\phi$

$r\,d\theta$ $r\,d\theta$

$r\sin(\theta+d\theta)\,d\phi$

On curve-1, the polar angle is fixed at $\theta + d\theta$ and the length is $r \sin(\theta + d\theta)\, d\phi$, whereas on curve-3 the angle and length are θ and $r \sin\theta$ respectively. On curves-4 and -2, the azimuthal angles are fixed as ϕ and $\phi + d\phi$ respectively, whereas both lengths are equal to $r\, d\theta$. The line integral of $\underline{F}$ is

$$\int_L \underline{F} \cdot \underline{d\ell} =$$

$$F_\phi(r, \theta + d\theta, \phi)\, r\, \sin(\theta + d\theta)\, d\phi - F_\theta(r, \theta, \phi + d\phi)\, r\, d\theta - F_\phi(r, \theta, \phi)\, r\, \sin\theta\, d\phi + F_\theta(r, \theta, \phi)\, r\, d\theta$$

$$\approx \left\{ \frac{\partial}{\partial\theta} \{\sin\theta\, F_\phi(r, \theta, \phi)\} - \frac{\partial}{\partial\phi} F_\phi(r, \theta, \phi) \right\} r\, d\theta\, d\phi\,. \tag{6.61}$$

From Eq. (6.44),

$$(\underline{\nabla} \times \underline{F}) \cdot \hat{S} = \lim_{S \to 0} \left\{ \frac{1}{S} \int_L \underline{F} \cdot \underline{d\ell} \right\},$$

so that

$$(\underline{\nabla} \times \underline{F})_r = \frac{1}{r\, \sin\theta} \left\{ \frac{\partial}{\partial\theta} \{\sin\theta\, F_\phi(r, \theta, \phi)\} - \frac{\partial}{\partial\phi} F_\theta(r, \theta, \phi) \right\}. \tag{6.62}$$

The other two components are evaluated in much the same way. The results are

$$(\underline{\nabla} \times \underline{F})_\theta = \frac{1}{r\, \sin\theta} \left\{ \frac{\partial}{\partial\phi} F_r(r, \theta, \phi) - \sin\theta\, \frac{\partial}{\partial r} \{r\, F_\phi(r, \theta, \phi)\} \right\}, \tag{6.63}$$

$$(\underline{\nabla} \times \underline{F})_\phi = \frac{1}{r} \left\{ \frac{\partial}{\partial r} \{r\, F_\theta(r, \theta, \phi)\} - \frac{\partial}{\partial\theta} F_r(r, \theta, \phi) \right\}. \tag{6.64}$$

Example

Evaluating the divergence of $\underline{F} = r\, \hat{\underline{e}}_\theta$ in spherical polar coordinates gives

$$\underline{\nabla} \cdot \underline{F} = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}(r \sin\theta) = \cot\theta\,.$$

In Cartesian coordinates,

$$\underline{F} = r \cos\theta \cos\phi\, \hat{\underline{e}}_x + r \cos\theta \sin\phi\, \hat{\underline{e}}_y - r \sin\theta\, \hat{\underline{e}}_z\,.$$

But

$$\cos\theta = \frac{z}{r}\,, \qquad \sin\theta = \frac{\sqrt{x^2 + y^2}}{r}\,,$$
$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}}\,, \qquad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}\,.$$

Thus

$$\underline{F} = \frac{zx}{\sqrt{x^2 + y^2}}\, \hat{\underline{e}}_x + \frac{zy}{\sqrt{x^2 + y^2}}\, \hat{\underline{e}}_y - \sqrt{x^2 + y^2}\, \hat{\underline{e}}_z\,.$$

$$\underline{\nabla} \cdot \underline{F} = \frac{z}{(x^2 + y^2)^{1/2}} - \frac{zx^2}{(x^2 + y^2)^{3/2}} + \frac{z}{(x^2 + y^2)^{1/2}} - \frac{zy^2}{(x^2 + y^2)^{3/2}} - 0$$
$$= \frac{z(x^2 + y^2)}{(x^2 + y^2)^{3/2}} = \frac{z}{\sqrt{x^2 + y^2}} = \cot\theta\,.$$

When working out the curl of $\underline{F}$, only the ϕ component survives and we are left with

$$\underline{\nabla} \times \underline{F} = 2\, \hat{\underline{e}}_\phi\,.$$

In Cartesian coordinates,

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{\underline{e}}_x & \hat{\underline{e}}_y & \hat{\underline{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{zx}{\sqrt{x^2+y^2}} & \frac{zy}{\sqrt{x^2+y^2}} & -\sqrt{x^2 + y^2} \end{vmatrix}$$

$$= -2 \frac{y}{\sqrt{x^2 + y^2}}\, \hat{\underline{e}}_x + 2 \frac{x}{\sqrt{x^2 + y^2}}\, \hat{\underline{e}}_y = -2 \sin\phi\, \hat{\underline{e}}_x + 2 \cos\phi\, \hat{\underline{e}}_y = 2\, \hat{\underline{e}}_\phi\,.$$

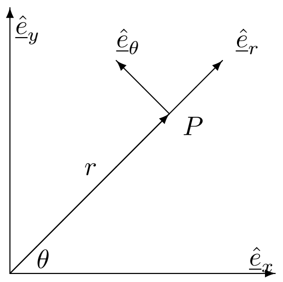
6.9 Cylindrical polar coordinates

The other coordinate system introduced in 1B21 is cylindrical polar coordinates, defined in terms of Cartesians by

$$\begin{aligned} x &= r\cos\theta\,, \\ y &= r\sin\theta\,, \\ z &= z\,. \end{aligned} \tag{6.65}$$

The basis vectors can be obtained geometrically; the z coordinate works just like a standard Cartesian coordinate and the polar part can be plot in a plane. $\underline{\hat{e}}_r$, $\underline{\hat{e}}_\theta$, and $\underline{\hat{e}}_z$ are defined as the directions in which the point P moves when the coordinates r and θ and z are increased by a very small amount.

$$\begin{aligned} \underline{\hat{e}}_r &= \cos\theta\,\underline{\hat{e}}_x + \sin\theta\,\underline{\hat{e}}_y\,, \\ \underline{\hat{e}}_\theta &= -\sin\theta\,\underline{\hat{e}}_x + \cos\theta\,\underline{\hat{e}}_y\,, \\ \underline{\hat{e}}_z &= \underline{\hat{e}}_z \end{aligned} \tag{6.66}$$



The basis vectors form an orthogonal system with

$$\underline{\hat{e}}_r \times \underline{\hat{e}}_\theta = \underline{\hat{e}}_z\,. \tag{6.67}$$

Note that r is used for the radius vector in the plane and cannot be used simultaneously for the position vector. For this, we shall use a capital;

$$\underline{R} = r\,\underline{\hat{e}}_r + z\,\underline{\hat{e}}_z\,, \tag{6.68}$$

When the coordinates change by infinitesimal amounts, the position vector changes

$$\underline{dR} = dr\,\underline{\hat{e}}_r + r\,d\theta\,\underline{\hat{e}}_\theta + dz\,\underline{\hat{e}}_z\,. \tag{6.69}$$

The volume element is the product of these three terms

$$dV = r\,dr\,d\theta\,dz\,, \tag{6.70}$$

and the elements of area pointing in the directions of the basis vectors

$$\begin{aligned} dS_r &= r\,d\theta\,dz\,, \\ dS_\theta &= dr\,dz\,, \\ dS_z &= r\,dr\,d\theta\,. \end{aligned} \tag{6.71}$$

The gradient in cylindrical polar coordinates is obtained from

$$df = ((\underline{\nabla}\,f)_r\,\underline{\hat{e}}_r + (\underline{\nabla}\,f)_\theta\,\underline{\hat{e}}_\theta + (\underline{\nabla}\,f)_z\,\underline{\hat{e}}_z)\cdot(dr\,\underline{\hat{e}}_r + r\,d\theta\,\underline{\hat{e}}_\theta + dz\,z) = (\underline{\nabla}\,f)_r\,dr + (\underline{\nabla}\,f)_\theta\,r\,d\theta + (\underline{\nabla}\,f)_z\,dz\,. \tag{6.72}$$

We can now read off the cylindrical polar components of the gradient using the chain rule for partial differentiation:

$$
\begin{aligned}
(\underline{\nabla} f)_r &= \frac{\partial f}{\partial r}\,, \\
(\underline{\nabla} f)_\theta &= \frac{1}{r}\frac{\partial f}{\partial \theta}\,, \\
(\underline{\nabla} f)_z &= \frac{\partial f}{\partial z}\,.
\end{aligned} \tag{6.73}
$$

To work out the divergence, we must first evaluate the flux through the hypercube. Proceeding as for the spherical polar coordinates, this is

$$
\begin{aligned}
&[F_r(r+dr,\theta,z)(r+dr) - F_r(r,\theta,z)r]\; d\theta\, dz \\
&\quad + [F_\theta(r,\theta+d\theta,z) - F_\theta(r,\theta,z)]\; dr\, dz \\
&\quad + [F_z(r,\theta,z+dz) - F_z(r,\theta,z)]\; r\, dr\, d\theta\,.
\end{aligned} \tag{6.74}
$$

Hence

$$
\int_S \underline{F}\cdot\underline{dS} \approx \left\{\frac{\partial}{\partial r}\left(r\,F_r(r,\theta,z)\right) + \frac{\partial}{\partial \theta}F_\theta(r,\theta,z) + r\,\frac{\partial}{\partial z}F_z(r,\theta,z)\right\} dr\, d\theta\, dz\,. \tag{6.75}
$$

Thus, by Eq. (6.44),

$$
\underline{\nabla}\cdot\underline{F} = \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\,F_r(r,\theta,z)\right) + \frac{\partial}{\partial \theta}F_\theta(r,\theta,z) + r\,\frac{\partial}{\partial z}F_z(r,\theta,z)\right\}\,. \tag{6.76}
$$

It is then straightforward to use Eqs. (6.73) and (6.76) together to get an expression for the Laplacian operator in cylindrical polar coordinates:

$$
\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}\,. \tag{6.77}
$$

To find the z-component of curl, look at the line integral around the elementary contour of the figure. On curve-1, the radial variable is fixed at $r + dr$ and the length is $(r + dr)\, d\theta$, whereas on curve-3 the radius and length are r and $r\, d\theta$ respectively. On curves-4 and -2, the angles are fixed as θ and $\theta + d\theta$ respectively, whereas both lengths are equal to dr. The line integral of $\underline{F}$ is

$$
\begin{aligned}
\int_L \underline{F}\cdot\underline{d\ell} &= F_\theta(r+dr,\theta,z)\,(r+dr)\,d\theta - F_r(r,\theta+d\theta,z)\,dr - F_\theta(r,\theta,z)\,r\,d\theta + F_r(r,\theta,z)\,dr \\
&\approx \left\{\frac{\partial}{\partial r}\,\{r\,F_\theta(r,\theta,\phi)\} - \frac{\partial}{\partial \theta}\,F_r(r,\theta,\phi)\right\} dr\, d\theta\,.
\end{aligned} \tag{6.78}
$$

$r \sin(\theta)\, d\phi$

$r\, d\theta$ $r\, d\theta$

$r \sin(\theta + d\theta)\, d\phi$

Hence

$$
(\underline{\nabla}\times\underline{F})_z = \frac{1}{r}\left\{\frac{\partial}{\partial r}\,\{r\,F_\theta(r,\theta,\phi)\} - \frac{\partial}{\partial \theta}\,F_r(r,\theta,\phi)\right\}\,. \tag{6.79}
$$

The other two components are a little easier to work out:

$$(\underline{\nabla} \times \underline{F})_r = \frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \,. \tag{6.80}$$

$$(\underline{\nabla} \times \underline{F})_\theta = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \,. \tag{6.81}$$